Completely generalized right primary rings and their extensions

Vijay Kumar Bhat

Abstract. A ring $R$ is said to be a completely generalized right primary ring (c.g.r.p ring) if $a, b \in R$ with $ab = 0$ implies that $a = 0$ or $b$ is nilpotent.

Let now $R$ be a ring and $\sigma$ an automorphism of $R$. In this paper we extend the property of a completely generalized right primary ring (c.g.r.p ring) to the skew polynomial ring $R[x; \sigma]$.

Key Words: Ore extension, automorphism, derivation, completely prime ideal

Mathematics Subject Classification 2010: 16N40, 16P40, 16S36.

Introduction

A ring $R$ means an associative ring with identity $1 \neq 0$. $\mathbb{Z}$ denotes the ring of integers and $\mathbb{N}$ denotes the set of positive integers unless otherwise stated.

This article concerns the study of skew polynomial rings over completely generalized right primary rings (c.g.r.p rings). Recall that a ring $R$ is said to be a c.g.r.p ring if $a; b \in R$ with $ab = 0$ implies that $a = 0$ or $b$ is nilpotent. An ideal $I$ of $R$ is said to be a completely generalized right primary ideal if $R/I$ is a completely generalized right primary ring ([9]).

Completely generalized left primary (c.g.l.p) rings and completely generalized left primary ideals are defined in a similar way.

Example (Example 2.2. of [9]) Let $A$ and $B$ be simple nil rings which are not nilpotent (for examples of such rings see Smoktunowicz [12]). Then $R = A \oplus B$ is a nil ring, and $R$ is completely g.r.p (g.l.p).

We now give a brief about the skew polynomial rings (also known as Ore extensions):

Let $R$ be a ring, $\sigma$ an automorphisms of $R$ and $\delta$ a $\sigma$-derivation of $R$; i.e. $\delta : R \rightarrow R$ is an additive mapping satisfying $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$.

For example let $\sigma$ be an automorphism of a ring $R$ and $\delta : R \rightarrow R$ any map.

Let $\phi : R \rightarrow M_2(R)$ be a map defined by
\[\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R.\]

Then \(\phi\) is a ring homomorphism if and only if \(\delta\) is a \(\sigma\)-derivation of \(R\).

We recall that the Ore extension

\[R[x; \sigma, \delta] = \{f = \sum_{i=0}^{n} x^i a_i, a_i \in R\}\]

with usual addition of polynomials and multiplication subject to the relation \(ax = x\sigma(a) + \delta(a)\) for all \(a \in R\). We denote \(R[x; \sigma, \delta]\) by \(O(R)\). If \(I\) is an ideal of \(R\) such that \(I\) is \(\sigma\)-stable (i.e. \(\sigma(I) = I\)) and is also \(\delta\)-invariant (i.e. \(\delta(I) \subseteq I\)), then clearly \(I[x; \sigma, \delta]\) is an ideal of \(O(R)\), and we denote it as usual by \(O(I)\).

In case \(\sigma\) is the identity map, we denote the differential operator ring \(R[x; \delta]\) by \(D(R)\). If \(J\) is an ideal of \(R\) such that \(J\) is \(\delta\)-invariant (i.e. \(\delta(J) \subseteq J\)), then clearly \(J[x; \delta]\) is an ideal of \(D(R)\), and we denote it by \(D(J)\). In case \(\delta\) is the zero map, we denote \(R[x; \sigma]\) by \(S(R)\). If \(K\) is an ideal of \(R\) such that \(K\) is \(\sigma\)-stable (i.e. \(\sigma(K) = K\)), then clearly \(K[x; \sigma]\) is an ideal of \(S(R)\), and we denote it by \(S(K)\).

The study of c.g.r.p (c.g.l.p) rings stems from Lasker-Noether concept of a primary ideal which has been extended to associative, not necessarily commutative rings. The concept of primary ideal in commutative rings has been generalized to a noncommutative setting by several authors, e.g., Barnes [1], Chatters and Hajarnavis [5], and Fuchs [7]. For more details on the concept of primary ideals and primary decomposition, the reader is referred to Noether [11] and Eisenbud [6].

A stronger type of primary decomposition (called transparency) for a right Noetherian ring has been introduced by the author of this paper in [2] as follows:

**Definition 1** A ring \(R\) is said to be an irreducible ring if the intersection of any two non-zero ideals of \(R\) is non-zero. An ideal \(I\) of \(R\) is called irreducible if \(I = J \cap K\) implies that either \(J = I\) or \(K = I\). Note that if \(I\) is an irreducible ideal of \(R\), then \(R/I\) is an irreducible ring.

**Proposition 1** Let \(R\) be a Noetherian ring. Then there exist irreducible ideals \(I_j, 1 \leq j \leq n\) of \(R\) such that \(\cap_{j=1}^{n} I_j = 0\).

**Proof.** The proof is obvious and we leave the details to the reader.\(\square\)

**Definition 2** (Definition 1.2 of [2]) A Noetherian ring \(R\) is said to be transparent ring if there exist irreducible ideals \(I_j, 1 \leq j \leq n\) such that \(\cap_{j=1}^{n} I_j = 0\) and each \(R/I_j\) has a right artinian quotient ring.
It can be easily seen that a Noetherian integral domain is a transparent ring, a commutative Noetherian ring is a transparent ring and so is a Noetherian ring having an artinian quotient ring. A fully bounded Noetherian ring is also a transparent ring.

The following result has been proved in Bhat [2] towards the transparency of skew polynomial rings.

Let $R$ be a commutative Noetherian ring and $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Then it is known that $S(R)$ and $D(R)$ are transparent. (Bhat [2])

The following result has been proved in Bhat [3].

**Theorem 1** (Theorem (3.4) of [3]) Let $R$ be a commutative Noetherian $\mathbb{Q}$-algebra, $\sigma$ an automorphism of $R$. Then there exists an integer $m \geq 1$ such that the skew polynomial ring $R[x; \alpha, \delta]$ is a transparent ring, where $\sigma^m = \alpha$ and $\delta$ is an $\alpha$-derivation of $R$ such that $\alpha(\delta(a)) = \delta(\alpha(a))$, for all $a \in R$.

**Completely generalized right primary ring:**

We now extend the notion of Completely generalized right primary rings to skew polynomial rings, and have the following:

**Definition 3** Let $R$ be a ring, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. We say that $O(R) = R[x; \sigma, \delta]$ is an extended completely generalized right primary ring (e.g.r.p ring) if for $f(x), g(x) \in O(R)$ (say $f(x) = \sum_{i=0}^{n} x^ia_i$ and $g(x) = \sum_{j=0}^{m} x^jb_j$), $f(x)g(x) = 0$ implies that $f(x) = 0$ or $b_j$ is nilpotent for all $j$, $0 \leq j \leq m$.

We prove the following in this direction:

**Theorem A:** Let $R$ be a c.g.r.p (c.g.l.p) ring and $\sigma$ an automorphism of $R$. Then $S(R) = R[x; \sigma]$ is an e.c.g.r.p (e.c.g.l.p) ring. This is proved in Theorem [2].

# 1 Completely generalized right primary rings and their extensions

We begin this section with the following:

Recall that an ideal $P$ of a ring $R$ is completely prime if $R/P$ is a domain, i.e. $ab \in P$ implies $a \in P$ or $b \in P$ for $a$, $b \in R$ (McCoy [10]).

In commutative case completely prime and prime have the same meaning. We also note that every completely prime ideal of a ring $R$ is a prime ideal, but the converse need not be true.
Example 1 Let $R = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{array} \right) = M_2(\mathbb{Z})$. If $p$ is a prime number, then the ideal $P = M_2(p\mathbb{Z})$ is a prime ideal of $R$, but is not completely prime, since for $a = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$ and $b = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$, we have $ab \in P$, even though $a \notin P$ and $b \notin P$.

Towards the completely prime ideals of $O(R)$, the following has been proved in [4]:

Theorem 2 (Theorem 2.4 of Bhat[4]): Let $R$ be a ring, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Then:

1. For any completely prime ideal $P$ of $R$ with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $O(P)$ is a completely prime ideal of $O(R)$.

2. For any completely prime ideal $U$ of $O(R)$, $U \cap R$ is a completely prime ideal of $R$.

Recall that an ideal $I$ of a ring $R$ is said to be completely semiprime if $a \in R$ such that $a^n \in I$ for some $n \in \mathbb{N}$ implies that $a \in I$ (McCoy [10]). With this we have the following known results:

Proposition 2 (Proposition 2.8. of [9]) Let $I$ be a completely semiprime ideal of $R$. Then $I$ is a completely prime ideal if and only if $I$ is a c.g.r.p (c.g.l.p) ideal of $R$.

Proposition 3 (Proposition 2.11. of [9]) Let $\phi : R \rightarrow R'$ be a surjective homomorphism and $I$ an ideal of $R$ such that $\text{Ker}(\phi) \subseteq I$. Then $I$ is a c.g.r.p (c.g.l.p) ideal of $R$ implies that $\phi(I)$ is a c.g.r.p (c.g.l.p) ideal of $R'$.

Proposition 4 (Proposition 2.12. of [9]) Let $\phi : R \rightarrow R'$ be a surjective homomorphism and $I'$ an ideal of $R'$ with $I' = \phi^{-1}(I)$. Then $I'$ is a c.g.r.p (c.g.l.p) in $R'$ implies $I$ is c.g.r.p (c.g.l.p) in $R$.

We now state and prove the main theorem of this article (regarding extended c.g.r.p rings) as follows:

Theorem 3 Let $R$ be a c.g.r.p (c.g.l.p) ring and $\sigma$ an automorphism of $R$. Then $S(R) = R[x; \sigma]$ is an e.c.g.r.p (e.c.g.l.p) ring.
Proof. We consider c.g.r.p case. The c.g.l.p shall follow on same lines.

Let $f(x); g(x) \in S(R)$ be such that $f(x)g(x) = 0$ (say $f(x) = \sum_{i=0}^{\infty} x^i a_i$, $g(x) = \sum_{i=0}^{m} x^i b_i$). We use induction on $m, n$ to prove the result. Let $m = n = 1$ say $f(x) = xa + b$, $g(x) = xc + d$.

Now $f(x)g(x) = 0$ implies that

$$x^2 \sigma(a)c + x(\sigma(b)c + ad) + bd = 0$$

This implies that

$$\sigma(a)c = 0, \sigma(b)c + ad = 0, bd = 0$$

Now $bd = 0$ implies that $b = 0$ or $d$ is nilpotent.

Now two cases arise:

1. $b = 0$

2. $b \neq 0$

(1) If $b = 0$, then $\sigma(b)c + ad = 0$ implies that $ad = 0$. Now $ad = 0$ implies that $a = 0$ or $d$ is nilpotent. If $a = 0$, then we have $f(x) = xa + b = 0$. If $a \neq 0$, then $d$ is nilpotent and $\sigma(a) \neq 0$. Therefore $\sigma(a)c = 0$ implies that $c$ is nilpotent. So we have $c, d$ are nilpotent.

(2) If $b \neq 0$, then $d$ is nilpotent. Now $\sigma(a)c = 0$ implies that $\sigma(a) = 0$ or $c$ is nilpotent. If $c$ is nilpotent, we have $c, d$ are nilpotent. If $c$ is non-nilpotent, then $\sigma(a) = 0$ or $a = 0$. Now $\sigma(b)c + ad = 0$ implies that $\sigma(b)c = 0$ and $c$ is non-nilpotent implies that $\sigma(b) = 0$ or $b = 0$. So $f(x) = xa + b = 0$.

Therefore, the result is true for $m = n = 1$.

Suppose the result is true for all polynomials $f(x); g(x)$ with $\deg(f(x)) = n$ and $\deg(g(x)) = m$.

We prove for $f(x); g(x)$ with $\deg(f(x)) = n + 1$ and $\deg(g(x)) = m + 1$. Let

$$f(x) = x^{n+1}c_{n+1} + \ldots + c_0, g(x) = x^{m+1}d_{m+1} + \ldots + d_0.$$ 

Now $f(x)g(x) = 0$ implies that

$$x^{m+n+2} \sigma^{m+1}(c_{n+1})d_{m+1} + x^{m+n+1}(\sigma^m(c_{n+1})d_m + \sigma^{m+1}(c_n)d_{m+1}) + \ldots + c_0d_0 = 0.$$

Now $\sigma^{m+1}(c_{n+1})d_{m+1} = 0$ implies that $\sigma^{m+1}(c_{n+1}) = 0$ or $d_{m+1}$ is nilpotent. Suppose $d_{m+1}$ is non-nilpotent, then $\sigma^m(c_{n+1}) = 0$ or $c_{n+1} = 0$. Also equating coefficient of $x^{m+n+1}$ to zero, we have $\sigma^m(c_{n+1})d_m + \sigma^{m+1}(c_n)d_{m+1} = 0$. Now $c_{n+1} = 0$ implies that $\sigma^{m+1}(c_n)d_{m+1} = 0$ and $d_{m+1}$ is non-nilpotent implies that $\sigma^{m+1}(c_n) = 0$ or $c_n = 0$.

Now equating coefficient of $x^{m+n}$ to zero, we get
Now \( c_{n+1} = c_n = 0 \) implies that \( \sigma^{m+1}(c_{n-1})d_{m+1} = 0 \) and \( d_{m+1} \) is non-nilpotent implies that \( \sigma^{m+1}(c_{n-1}) = 0 \) or \( c_{n-1} = 0 \). With the same process in a finite number of steps we get \( c_i = 0; \ 0 \leq i \leq n+1 \). Therefore, \( f(x) = 0 \).

\[ \sigma^{m-1}(c_{n+1})d_{m-1} + \sigma^m(c_n)d_m + \sigma^{m+1}(c_{n-1})d_{m+1} = 0. \]

\[ \text{Remark 1} \quad \text{We have not been able to prove the result for } O(R) = R[x; \sigma, \delta], \text{ where } \sigma \text{ is an automorphism of } R \text{ and } \delta \text{ is a } \sigma \text{-derivation of } R. \]

Let \( f(x) = xa + b \), \( g(x) = xc + d \).

Now \( f(x)g(x) = 0 \) implies that

\[ x^2\sigma(a)c + x(\delta(a)c + \sigma(b)c + ad) + \delta(b)c + bd = 0. \]

So we have

\[ \sigma(a)c = 0, \ \delta(a)c + \sigma(b)c + ad = 0, \ \delta(b)c + bd = 0 \]

Now \( \sigma(a)c = 0 \) implies that \( \sigma(a) = 0 \) or \( c \) is nilpotent.

If \( \sigma(a) = 0 \), i.e. \( a = 0 \), then \( \delta(a)c + \sigma(b)c + ad = 0 \) implies that \( \sigma(b)c = 0 \).

Therefore \( \sigma(b) = 0 \) or \( c \) is nilpotent. If \( \sigma(b) = 0 \), i.e. \( b = 0 \), then we have \( f(x) = 0 \).

If \( \sigma(b) \neq 0 \), then \( c \) is nilpotent and \( \delta(b)c + bd = 0 \) gives nothing about the nilpotency of \( d \).

References


Vijay Kumar Bhat

*Department of Mathematics, SMVD University, Katra, India*

*vijaykumarbhat2000@yahoo.com*

Please, cite to this paper as published in Armen. J. Math., V. 9, N. 1(2017), pp. 20-26