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On the action of differentiation operator in some classes of Nevanlinna-Djrbashian type in the unit disk and polydisk

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Abstract

We introduce new area Nevanlinna type spaces in the unit disk and polydisk and study the action of classical operator of differentiation on them. We substantially complement the list of previously known assertions of this type.

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$, $\mathrm{T} = \{|z| = 1\}$ as unit circle, $I^n = (0,1]^n, \mathrm{T}^n = \mathrm{T} \times \cdots \times \mathrm{T}$, $\mathbb{D}^n = \{z = (z_1, z_2, \cdots, z_n) : |z_j| < 1, j = 1, 2, \cdots, n\}$ as unit polydisk, $H(\mathbb{D})$ be the space of all holomorphic functions in the unit disk, and let $H(\mathbb{D}^n)$ be the space of all holomorphic functions in the polydisk. Let $T(\tau, f)$ be Nevanlinna characteristic of f, $f \in H(\mathbb{D})$ (see [5]). Let below always w be a function from a set of all positive growing functions, $w \in L^1(0,1)$ such that there are two numbers $m_w > 0, M_w > 0$ and a number $q_w \in (0,1)$ such that $m_w \le \frac{w(\lambda\tau)}{w(\tau)} < M_w$, $\tau \in (0,1)$, $\lambda \in [q_w, 1]$ (see [7]). Let $w \in S$ then there are measurable functions $\varepsilon(x), q(x)$ so that

$$w(x) = \exp\left\{ q(x) + \int_x^1 \frac{\varepsilon(u)}{u} du \right\},\ x \in (0,1)$$

(see [7]). This characterization gives various examples of functions from S class. A typical example is $w(r)=r^{\alpha}, \alpha>-1, r\in(0,1)$ or $w(r)=r^{\alpha}(\log\frac{C}{r})^{\beta}, \alpha>-1, \beta>0, r\in(0,1), C$ is a constant.

We define several subspaces of $H(\mathbb{D})$ for fixed function $w\in L^1(0,1]$, $w>0$.

$$N^1_{p,w,\beta}=\{f\in H(\mathbb{D}):\sup_{0<R\le 1}\int_0^R (T(\tau,f))^p w(1-\tau)d\tau(1-R)^{\beta}<+\infty\},$$
$$N^2_{p,w,\alpha}=\{f\in H(\mathbb{D}):\int_0^1\left[\sup_{\tau\in(0,R]}(T(\tau,f))^p w(1-\tau)\right](1-R)^{\alpha}dR<+\infty\},$$
$$N^3_{p,q,w,\alpha}=\{f\in H(\mathbb{D}):\int_0^1\left(\int_0^R (T(\tau,f))^p w(1-\tau)d\tau\right)^{\frac{q}{p}}(1-R)^{\alpha}dR<+\infty\},$$
$$N^4_{p,q,w}=\{f\in H(\mathbb{D}):\int_0^1\left(\int_{-\pi}^{\pi}(\ln^+|f(\tau\xi)|)^p d\xi\right)^{\frac{q}{p}}w(1-\tau)d\tau<+\infty\},$$
$$N^5_{p,q,w}=\{f\in H(\mathbb{D}):\int_{-\pi}^{\pi}\left(\int_0^1(\ln^+|f(\tau\xi)|)^p w(1-\tau)d\tau\right)^{\frac{q}{p}}d\xi<+\infty\},$$
$$N^p=\{f\in H(\mathbb{D}):\sup_{\tau<1}\int_{-\pi}^{\pi}(\ln^+|f(\tau\xi)|)^p d\xi<\infty\}$$

where $0<p,q<\infty$, $\alpha>-1$, $\beta\ge 0$.

Note that these are complete metric spaces which can be checked without difficulties.

It is obvious that for $q=\infty$, $w=1$ the $N^4_{p,q,w}$ coincides with well-known N^p spaces of holomorphic functions with bounded characteristic, see [1].

In recent papers ([1, 8]), it was noted that the following assertions concerning the action of differentiation $\mathcal{D}(f)(z)=f'(z)$ and integration $I(f)(z)=\int_0^z f(t)dt$ are valid in mentioned analytic classes. $N^4_{q,q,\alpha}$ is closed under differentiation and integration operator(if $w(|z|)=(1-|z|)^{\alpha}$ we denote $N^4_{p,q,w}$ by $N^4_{p,q,\alpha}$) $N^4_{q,q,w}$ and $N^4_{1,q,w}$ are closed under differentiation operator $\mathcal{D}(f)$ if and only if $\int_0^1 w(t)(\ln\frac{1}{t})^p dt<+\infty$. The study $I(f),\mathcal{D}(f)$ in Smirnov N^+ class were studied also before(see [10] and references there).

We note much earlier in [2] Frostman then W. K. Hayman ([4]) established that the N^p class is not invariant under differentiation operator, but $N^p, p>1$ are closed for integration operator, but not N^1.

The natural question is to study differentiation operator in $N^i_{p,w,\alpha}$, $i=1,2,3,4,5$. The goal of this paper is to provide several new sharp results in this direction. Finally we would like to indicate that all assertions of this note were obtained by modification of approaches and arguments provided recently in [8]. All our results in higher dimension were obtained for $n=1$ in [8]. Throughout the paper, we write C (sometimes with indices) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

2 Main results

Motivated by mentioned results in this section we provide new assertions concerning differentiation operator $\mathcal{D}(f)$ in new Nevanlinna-Djrbashian type spaces that were defined above. In the following assertion, we provide several sharp results on the action of the differentiation operator in Nevanlinna type analytic spaces in the unit disk complementing previously known propositions of this type obtained before by various authors (see for example, [2]-[10] and references there).

Theorem 1. *1) $N^1_{p,w,\alpha}$ is closed under differentiation operator $\mathcal{D}(f)$ if and only if*

$$\sup_{R\in(0,1)}(1-R)^{\alpha}\int_0^R\left(\ln\frac{1}{1-\tau}\right)^p w(1-\tau)d\tau<\infty,\ 0<p<\infty,\ \alpha\geq 0.$$

2) $N^2_{p,w,\alpha}$ is closed under differentiation operator $\mathcal{D}(f)$ if and only if

$$\int_0^1\sup_{R<\tau}w(1-R)\left(\ln\frac{1}{1-R}\right)^p(1-\tau)^{\alpha}d\tau<\infty,\ 0<p<\infty,\ \alpha>-1.$$

3) $N^3_{p,q,w,\alpha}$ is closed under differentiation operator $\mathcal{D}(f)$ if and only if

$$\int_0^1\left(\int_0^R w(1-\tau)\left(\ln\frac{1}{1-\tau}\right)^p d\tau\right)^{\frac{q}{p}}(1-R)^{\alpha}dR<\infty,\ 0<p,q<\infty,\ \alpha>-1.$$

In the following theorem we provide sharp assertions concerning the operator of differentiation in $N^4_{p,q,\widetilde{w}}$ and $N^5_{p,q,\widetilde{w}}$.

Theorem 2. *$\mathcal{D}(f)$ is acting from $N^4_{p,q,\widetilde{w}}$ and $N^5_{p,q,\widetilde{w}}$ to $N^1_{s,s,w}$,*

$$\widetilde{w}(1-|z|)=w(1-|z|)^{\frac{q}{s}}(1-|z|)^{\frac{2q}{s}-\frac{q}{p}-1},\ \frac{2}{s}-\frac{1}{p}>0,\ s\geq 1,\ s\geq\max\{q,p\}$$

if and only if

$$\int_0^1\left(\ln\frac{1}{t}\right)^s w(t)dt<\infty.$$

Now we formulate some new sharp results in higher dimensions. Let always below for any function $f\in H(\mathbb{D}^n)$,

$$\mathcal{D}f(z)=\frac{\partial f(z_1,z_2,\dots,z_n)}{\partial z_1,\dots,\partial z_n}.$$

Note Nevanlinna type classes in higher dimension were studied also before see for example [6] and references there.

Theorem 3. *Let* $0 < p < \infty$, $\int_0^1 w_j(t)dt < +\infty$, $j = 1, 2, \ldots, n$. *Then*

$$\int_{I^n}\left(\int_{\mathrm{T}^n}\ln^+\left|\mathcal{D}f(\tau_1\xi_1,\ldots,\tau_n\xi_n)|d\xi_1\ldots d\xi_n\right)^p\prod_{j=1}^n w_j(1-\tau_i)d\tau_1\ldots d\tau_n$$
$$\leq C\int_{I^n}\left(\int_{\mathrm{T}^n}\ln^+|f(\tau_1\xi_1,\ldots,\tau_n\xi_n)|d\xi_1\ldots d\xi_n\right)^p\prod_{j=1}^n w_j(1-\tau_i)d\tau_1\ldots d\tau_n.$$

if and only if

$$\int_0^1 w_j(t)\left(\ln\frac{1}{t}\right)^p dt < +\infty,\ j = 1, 2, \ldots, n.$$

Theorem 4. *Let* $s \geq 1, s \geq \max\{q, p\}, w = \Pi_{j=1}^n w_j$. *Let*

$$\frac{2}{s} - \frac{1}{p} > 0,\ \widetilde{w_j}(1-|z_j|) = w_j(1-|z_j|)^{\frac{q}{s}}(1-|z_j|)^{\frac{2q}{s}-\frac{q}{p}-1}.$$

Then $\mathcal{D}f$ *is acting from* $N^4_{p,q,\widetilde{w}}(N^5_{p,q,\widetilde{w}})$ *to* $N^1_{s,s,w}$ *if and only if*

$$\int_0^1 w_j(1-\tau)\left(\ln\frac{1}{1-\tau}\right)^s d\tau < +\infty,\ j = 1, 2, \ldots, n,$$

where

$$N^5_{p,q,w}(\mathbb{D}^n) = \{f \in H(\mathbb{D}^n) : \int_{\mathrm{T}^n}\left(\int_{I^n}(\ln^+|f(\tau\xi)|)^p\prod_{k=1}^n w(1-\tau_k)d\tau_1\ldots d\tau_n\right)^{\frac{q}{p}}d\xi < +\infty\},$$
$$N^4_{p,q,w}(\mathbb{D}^n) = \{f \in H(\mathbb{D}^n) : \int_{I^n}\left(\int_{\mathrm{T}^n}(\ln^+|f(\tau\xi)|)^p d\xi\right)^{\frac{q}{p}}\prod_{k=1}^n w(1-\tau_k)d\tau_1\ldots d\tau_n < +\infty\}.$$

Let us mention some lemmas that are needed for the proofs.

Lemma A([9]). *For* $f \in H(\mathbb{D}^n), s \geq \max(p, q), s > 1$. *Then*

$$\int_{\mathbb{D}^n}(\ln^+|f(z)|)^s\prod_{k=1}^n w(1-|z_k|)dm_{2n}(z)$$
$$\leq C\int_{I^n}\prod_{k=1}^n(w(1-|z_k|))^{\frac{q}{s}}(1-|z_k|)^{\frac{2q}{s}-\frac{q}{p}-1}\left(\int_{T^n}\log^+|f(z)|^q dm_n\xi\right)^{\frac{q}{p}}d|z|$$
$$\int_{\mathbb{D}^n}(\ln^+|f(z)|)^s\prod_{k=1}^n w(1-|z_k|)dm_{2n}(z)$$
$$\leq C\int_{T^n}\left(\int_{I^n}\prod_{k=1}^n(w(1-|z_k|))^{\frac{q}{s}}(1-|z_k|)^{\frac{2q}{s}-\frac{q}{p}-1}\log^+|f(z)|d|z|\right)^{\frac{q}{p}}dm_n\xi.$$

Lemma 1. *([3]). The following estimates are true.*
1)

$$\int_{\mathbb{T}^n} \ln^+ |\mathcal{D}f(\tau_1\varphi_1, \dots, \tau_n\varphi_n)| d\varphi_1 \dots d\varphi_n$$
$$\leq C\Bigg(\Bigg(\sum_{j=1}^{n} \ln \frac{1}{1-\tau_j}\Bigg) + \int_{\mathbb{T}^n} \ln^+ |f(\overrightarrow{\tau}\xi)| dm_n(\xi)\Bigg)$$
$$\overrightarrow{\tau} = \Bigg(\frac{1+\tau_1}{2}, \dots, \frac{1+\tau_n}{2}\Bigg),\ \tau_i \in (0,1), i = 1, \dots, n;$$

2)

$$\ln^+ T\Bigg(\frac{1+\tau}{2}, f\Bigg) \leq CT\Bigg(\frac{1+\tau}{2}, f\Bigg), \tau \in (0,1),$$
$$T(R, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln^+ |f(R\xi)| d\xi, R \in (0,1).$$

Lemma 2. *([8]). Let $\lambda_k = 2^{\lambda k}$, $\lambda > 0$, $\tau_n = \exp(-\frac{1}{2^{n\lambda}})$. Then for $\varphi \in [0, 2\pi]$, there exists a function $f, f \in H(\mathbb{D})$,*

$$\ln^+ |f'(\tau_n e^{i\varphi})| \geq C \ln \frac{1}{1-\tau_n}, f(z) = \sum_{k=0}^{\infty} \lambda_k^{\alpha-1} z^{\lambda_k}, 0 < \alpha < 1, \lambda > 0.$$

We formulate the following assertion since it is interesting by itself and it is a core of the proof of necessity part of our theorems in polydisk (see section 4).

Lemma 3. *1) Let $R_{m_j} = \exp(-\frac{1}{2^{\lambda m_j}}) \in (0,1]$, $t \in (0, +\infty)$, $\lambda > 0$, $j = 1, 2, \dots, n$. Then there exists a function $f, f \in H(\mathbb{D}^n)$,*

$$\Bigg(\ln^+ |\mathcal{D}f(R_{m_1}e^{i\varphi_1}, \dots, R_{m_n}e^{i\varphi_n})|\Bigg)^t \geq C \sum_{j=0}^{n} \Bigg(\ln \frac{1}{1-R_{m_j}}\Bigg)^t,\ \varphi_i \in (0, 2\pi].$$

2)

$$\int_{\mathbb{T}^n} \Bigg(\ln^+ |\mathcal{D}f(\tau_1\xi_1, \dots, \tau_n\xi_n)|\Bigg)^s d\xi_1 \dots d\xi_n$$

is growing as a function of $\tau_1, \dots, \tau_n$ for every $s \geq 1$, $f \in H(\mathbb{D}^n)$.

Remark 1. *The statements in the Theorem 2 for $q = p = s$ were established in [8].*

Remark 2. *As W. Hayman shows in the unit disk there is a function so that $T(\tau, I(f)) > C \ln \frac{1}{1-\tau}$, $T(\tau, f) < C$, $\tau \in (0,1)$. Let $X(w)$ be any class of functions with $\|\cdot\|_{X(w)}$ quasi-norm so that $(N^1) \subset X(w), X(w) \subset H(\mathbb{D})$. If for any $g \in X(w), I(g) \in X(w)$, then clearly for Hayman's function it is also true. Hence $I(f) \in X(w), f \in X(w)$, $T(\tau, I(f)) > C \ln \frac{1}{1-\tau}, \tau \in (0,1)$. Hence if $(X(w)) \subset X_1$, $X_1 = \{f \in H(\mathbb{D}) : \|T(\tau, f)\|_{Y(w,[0,1))} < \infty\}$, then $\| \ln \frac{1}{1-\tau} \|_{Y(w,[0,1))} < \infty$. As $X(w)$ obviously we can take any space $N^i_{p,q,w}, i = 1, 2, 3, 4, 5$ under some natural additional assumption on w.*

Remark 3. *It is not difficult to see that the statements of Theorem 1 and Theorem 2 remain true if we replace $\mathcal{D}$ operator by $\bigwedge(f)(z) = \sum_{k=0}^{n} f_k(z)\mathcal{D}^k(f)(z)$, where f_k are functions from $N^i_{p,q,w}$, $i = 1, 2, 3, 4, 5$.*

Note that with the help of so-called slice functions technique in [8, 11], some results of this paper can be even expanded to the unit ball.

3 Proofs of Theorems 1-4 (sufficiency of conditions)

The following estimate can be found in [3] and [8].

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^+ |f'(\tau e^{i\varphi})| d\varphi \le C\Big(\ln T(R, f) + T(\tau, f) + \ln \frac{1}{1-\tau}\Big),$$

where $f \in H(\mathbb{D})$, and $\tau < R < 1$, and $T(R, f)$ is as above a Nevanlinna characteristic of f. From last estimate and second estimate in Lemma 1 we have putting $R = \frac{1+\tau}{2}$ the following estimates

$$\begin{aligned}
&\int_0^R (T(\tau, f'))^p w(1-\tau) d\tau (1-R)^\beta \\
&\quad \le C \int_0^R \left(T(\frac{1+\tau}{2}, f)\right)^p w(1-\tau) d\tau (1-R)^\beta + C \int_0^R (T(\tau, f))^p w(1-\tau) d\tau (1-R)^\beta \\
&\qquad + C \int_0^R \left(\ln \frac{1}{1-\tau}\right)^p w(1-\tau) d\tau (1-R)^\beta, \beta \ge 0, 0 < p < \infty.
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \left(\sup_{\tau\in(0,R]} (T(\tau, f'))^p w(1-\tau)\right)(1-R)^\alpha dR \\
&\quad \le C \int_0^1 \sup_{\tau\in(0,R]} (T(\tau, f))^p w(1-\tau)(1-R)^\alpha dR \\
&\quad + C \int_0^1 \sup_{\tau\in(0,R]} \left(T\left(\frac{1+\tau}{2}, f\right)\right)^p w(1-\tau)(1-R)^\alpha dR \\
&\quad + C \int_0^1 \sup_{\tau\in(0,R]} \left(\ln \frac{1}{1-\tau}\right)^p w(1-\tau)(1-R)^\alpha dR
\end{aligned}$$

and finally

$$\begin{aligned}
&\int_0^1 \left(\int_0^R (T(\tau, f'))^p w(1-\tau) d\tau\right)^{\frac{q}{p}} (1-R)^\alpha dR \\
&\quad \le C \int_0^1 \left(\int_0^R (T(\tau, f))^p w(1-\tau) d\tau\right)^{\frac{q}{p}} (1-R)^\alpha dR \\
&\quad + C \int_0^1 \left(\int_0^R \left(T\left(\frac{1+\tau}{2}, f\right)\right)^p w(1-\tau) d\tau\right)^{\frac{q}{p}} (1-R)^\alpha dR \\
&\quad + C \int_0^1 \left(\int_0^R \left(\ln \frac{1}{1-\tau}\right)^p w(1-\tau) d\tau\right)^{\frac{q}{p}} (1-R)^\alpha dR
\end{aligned}$$

To finish one side of the proof in Theorem 1 it remains to show that

$$\sup_{R\le 1}\left(\int_0^R\left(T\left(\frac{1+\tau}{2},f\right)\right)^p w(1-\tau)d\tau\right)(1-R)^\beta \le C\|f\|_{N^1_{p,w,\beta}}$$

$$\int_0^1\left(\sup_{\tau\in(0,R]}\left(T\left(\frac{1+\tau}{2},f\right)\right)^p w(1-\tau)\right)(1-R)^\alpha \le C\|f\|_{N^2_{p,w,\alpha}}$$

and finally

$$\int_0^1\left(\int_0^R\left(T\left(\frac{1+\tau}{2},f\right)\right)^p w(1-\tau)d\tau\right)^{\frac{q}{p}}(1-R)^\alpha dR \le C\|f\|_{N^3_{p,q,w,\alpha}}.$$

Let us show the first and the second estimate. The last estimate can be shown similarly by standard change of variables. We have the following inequalities

$$\begin{aligned}\sup_{0\le R\le 1}&\left(\int_0^R\left(T\left(\frac{1+\tau}{2},f\right)\right)^p w(1-\tau)d\tau\right)(1-R)^\beta\\ &\le C\sup_{0\le R\le 1}\left(\int_{\frac{1}{2}}^{\frac{1+R}{2}}\Big(T(u,f)\Big)^p w(2(1-u))du\right)(1-R)^\beta\\ &\le C\sup_{\frac{1}{2}\le t\le 1}\left(\int_{\frac{1}{2}}^{t}\Big(T(u,f)\Big)^p w(2(1-u))du\right)(1-t)^\beta \le C\|f\|_{N^1_{p,w,\beta}}\end{aligned}$$

We also have

$$\begin{aligned}\int_0^1&\sup_{\tau\in(0,R]}T\left(\frac{1+\tau}{2},f\right)w(1-\tau)(1-R)^\alpha dR\\ &\le C\int_0^1\sup_{t\in(\frac{1}{2},\frac{1+R}{2})}T(t,f)w(2(1-t))(1-R)^\alpha dR\\ &\le C\int_0^1\sup_{t\in(0,v)}T(t,f)w(1-t)(1-v)^\alpha dv.\end{aligned}$$

Above we used the fact that $w\in S$ and $\frac{w(\lambda\tau)}{w(\tau)}\in[m_w,M_w],\tau\in(0,1),\lambda\in[q_w,1]$.

For proof of sufficiency of condition in Theorem 2 we act in a usual way using Cauchy formula. For estimates of derivatives of f we have by Cauchy formula

$$f(z)=\int_{K_\eta(z)}\frac{f(\xi)d\xi}{\xi-z};K_\eta(z)=\{\xi:|\xi-z|=\eta(1-|z|)\}.$$

Then $f'(z)=\int_{K_\eta(z)}\frac{f(\xi)d\xi}{(\xi-z)^2}$ and $|f'(z)|\le C\frac{\max_{\xi\in K_\eta(z)}|f(\xi)|}{(1-|z|)}$. Hence we have that for $s\ge 1$

$$(\ln^+|f'(z)|)^s\le C\left(\max_{\xi\in K_\eta(z)}(\ln^+|f(\xi)|)^s+\left(\ln\frac{1}{1-|z|}\right)^s\right). \tag{1}$$

On the other hand using standard dyadic decomposition of unit disk we have for $s\ge 1$

$$\int_{\mathbb{D}} w(1-|w|)(\ln^+|f'(w)|)^s dm_2(w) \le C\sum_{k=0}^{\infty}\sum_{l=2^{-k}}^{2^k-1} \max_{\triangle_{k,l}}(\ln^+|f(z)|)^s w(|\triangle_{k,l}|^{\frac{1}{2}})|\triangle_{k,l}|.$$

Taking η so small that estimate (1) can be applied in last inequality we will have

$$\begin{aligned}
&\int_{\mathbb{D}} w(1-\tau)(\ln^+|f'(w)|)^s dm_2(w)\\
&\qquad\le C\int_0^1\int_{-\pi}^{\pi} w(1-\tau)(\ln^+|f(\tau e^{i\varphi})|)^s \tau d\tau d\varphi\\
&\qquad\qquad + C\int_0^1 w(1-\tau)\left(\ln^+\frac{1}{1-\tau}\right)^s d\tau, \quad s\ge 1.
\end{aligned}$$

Now using the fact that for $s\ge 1$, $(\ln^+|f(\tau e^{i\varphi})|)^s$ is subharmonic we estimate again using Lemma A for $n=1$ the following expression to get what we need.

$$\int_{-\pi}^{\pi}\int_0^1 w(1-\tau)(\ln^+|f(\tau e^{i\varphi})|)^s \tau d\tau d\varphi.$$

We turn to Theorem 3.

As we noted above if $f\in H(\mathbb{D})$ and $R\in(\tau,1)$, then we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln^+|f(\tau e^{i\varphi})|d\varphi\\
\le\ & C\Big(\ln T(R,f)+T(\tau,f)+\ln\frac{1}{1-\tau}\Big)\\
\le\ & C\Big(T(R,f)+T(\tau,f)+\ln\frac{1}{1-\tau}\Big).
\end{aligned}$$

So finally we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln^+|f(\tau e^{i\varphi})|d\varphi \le C\Big(T(R,f)+T(\tau,f)+\ln\frac{1}{1-\tau}\Big).$$

And from here we can get using this last formula by each variable separately and putting $R_j=\frac{1+\tau_j}{2}, j=1,2,\ldots,n$

$$\begin{aligned}
&\left(\frac{1}{2\pi}\right)^n\int_{T^n}\ln^+|\mathcal{D}f(\tau_1 e^{i\varphi_1},\ldots,\tau_n e^{i\varphi_n})|d\varphi_1\ldots d\varphi_n\\
\le\ & C\Big(\sum_{j=1}^{n}\ln\frac{1}{1-\tau_j}+\int_{T^n}\log^+\left|f\left(\frac{1+\tau}{2}\xi\right)\right|d\xi_1\ldots d\xi_n\\
&+\int_{T^n}\log^+|f(\tau\xi)|d\xi_1\ldots d\xi_n\Big) \qquad (2)
\end{aligned}$$

where
$$\frac{1+\tau}{2}\xi=\left(\frac{1+\tau_1}{2}\xi_1,\dots,\frac{1+\tau_n}{2}\xi_n\right),\tau_j\in(0,1),\xi_j\in T=\{|z|=1\}.$$

Let us show the last formula in case of bidisk (for $n=2$). The general case can be considered similarly we have

$$\begin{aligned}
&\left(\frac{1}{2\pi}\right)^2\int_{T^2}\ln^+\left|\frac{\partial^2 f}{\partial z_1\partial z_2}(\tau_1e^{i\varphi_1},\dots,\tau_ne^{i\varphi_n})\right|d\varphi_1d\varphi_2\\
\leq\ & C\Bigg(\ln\frac{1}{1-\tau_1}+\int_{T^2}\log^+\left|\frac{\partial f}{\partial z_2}\left(\frac{1+\tau_1}{2}\xi_1,\tau_2\xi_2\right)\right|d\xi_1d\xi_2\\
&+\int_{T^2}\log^+\left|\frac{\partial f}{\partial z_2}(\tau_1\xi_1,\tau_2\xi_2)\right|d\xi_1d\xi_2\Bigg)\\
\leq\ & C\Bigg(\ln\frac{1}{1-\tau_1}+\int_{T^2}\log^+|f(\tau_1\xi_1,\tau_2\xi_2)|d\xi_1d\xi_2\\
&+2\ln\frac{1}{1-\tau_2}+\int_{T^2}\log^+\left|f\left(\tau_1\xi_1,\frac{1+\tau_2}{2}\xi_2\right)\right|d\xi_1d\xi_2\\
&+\int_{T^2}\log^+\left|f\left(\frac{1+\tau_1}{2}\xi_1,\tau_2\xi_2\right)\right|d\xi_1d\xi_2\\
&+\int_{T^2}\log^+\left|f\left(\frac{1+\tau_1}{2}\xi_1,\frac{1+\tau_2}{2}\xi_2\right)\right|d\xi_1d\xi_2\Bigg)\\
\leq\ & C_1\left(\ln\frac{1}{1-\tau_1}+\ln\frac{1}{1-\tau_2}\right)\\
&+C_2\int_{T^2}\log^+\left|f\left(\tau_1\xi_1,\tau_2\xi_2\right)\right|d\xi_1d\xi_2\\
&+C_3\int_{T^2}\log^+|f(\frac{1+\tau_1}{2}\xi_1,\frac{1+\tau_2}{2}\xi_2)|d\xi_1d\xi_2.
\end{aligned}$$

The general case can be done similarly. We have used the fact that $(\log^+|f|)^s$ is subharmonic for $s\geq 1, f\in H(\mathbb{D})$. From here we have using properties of functions from these classes and change of variables one part of Theorem 3 the sufficiency.

Remark Putting in (2),$\tau_j=\tau\in(0,1), j=1,2,\dots,n$ we get immediately also the following assertion if $\int_0^1 w(t)\ln\frac{1}{t}dt<+\infty, w\in S$, then

$$\begin{aligned}
&\int_0^1\left(\int_{T^n}\ln^+|\mathcal{D}f(\tau_1e^{i\varphi_1},\dots,\tau_ne^{i\varphi_n})|d\varphi_1\dots d\varphi_n\right)^p w(1-\tau)d\tau\\
\leq\ & C\int_0^1\left(\int_{T^n}\ln^+|f(\tau_1e^{i\varphi_1},\dots,\tau_ne^{i\varphi_n})|d\varphi_1\dots d\varphi_n\right)^p w(1-\tau)d\tau, 0<p<\infty.
\end{aligned}$$

Now we turn to sufficiency of Theorem 4. First we note that Theorem 4 for $n=1$ case of unit disk completely coincide with Theorem 2 and we will only provide some sketch of the proof of Theorem 4 (sufficiency part), we will argue as follows in case of polydisk. First we will use the well-known Cauchy formula for polydisks and the dyadic decomposition of the polydisk to generalize the corresponding estimates that were obtained above for the case of unit disk, we will have the following.

$$f(z_1,\dots,z_n)=C\int_{K_\eta(z)}\frac{f(\xi)}{\xi-z}d\xi,\xi-z=\prod_{k=1}^{n}(\xi_k-z_k),d\xi=d\xi_1\dots d\xi_n,$$

$$f(\xi)=f(\xi_1,\dots,\xi_n),K_\eta(z)=K_{\eta_1}(z_1)\times\dots K_{\eta_n}(z_n),$$

$$K_{\eta_i}(z_i)=\{\xi_i\in\mathbb{D}:|\xi_i-z_i|=\eta_i(1-|z_i|)\},z\in\mathbb{D}^n,\eta\in\mathbb{R}^n_+.$$

Hence

$$\frac{\partial f(z_1,\dots,z_n)}{\partial z_1\dots\partial z_n}\le C\max_{\xi\in K_\eta(z)}|f(\xi_1,\dots,\xi_n)|\frac{1}{\prod_{k=1}^{n}(1-|z_k|)}.$$

Hence

$$\begin{aligned}&(\ln^+|\mathcal{D}f(z_1,\dots,z_n)|)^{\widetilde{s}}\\ \le\ & C\max_{\xi\in K_\eta(z)}(\ln^+|f(\xi_1,\dots,\xi_n)|)^{\widetilde{s}}+\left(\ln^+\frac{1}{1-|z_1|}\right)^{\widetilde{s}}\\ &+\dots+\left(\ln^+\frac{1}{1-|z_n|}\right)^{\widetilde{s}},\widetilde{s}\in(0,\infty)\end{aligned}$$

Note form above we have

$$\begin{aligned}&\int_{\mathbb{D}^n}(\ln^+|\mathcal{D}f(z)|)^s\prod_{k=1}^{n}w(1-|z_k|)dm_{2n}(z_1,\dots,z_n)\\ \le\ & C\sum_{k_1=1}^{\infty}\dots\sum_{k_n=1}^{\infty}\sum_{l_1=-2^{k_1}}^{2^{k_1}-1}\dots\sum_{l_1=-2^{k_n}}^{2^{k_n}-1}\left[\max_{\xi\in U_{\vec{k},\vec{l}}}(\ln^+|f(\xi)|)^s+\sum_{j=1}^{n}\left(\ln\frac{1}{1-|z_j|}\right)^s\right]\\ &\prod_{j=1}^{n}w_j(|U_{\vec{k},\vec{l}}|^{\frac{1}{2}})|U_{\vec{k},\vec{l}}|\end{aligned}$$

where

$$|U_{\vec{k},\vec{l}}|=\prod_{j=1}^{n}|U_{k_j,l_j}|,\vec{k}=(k_1,\dots,k_n),\vec{l}=(l_1,\dots,l_n),$$

$$U_{k_j,l_j}=\{z:\frac{\pi l_j}{2^{k_j}}\le argz\le\frac{\pi l_j+1}{2^{k_j}},1-\frac{1}{2^{k_j}}<|z|\le1-\frac{1}{2^{k_j+1}}\},j=1,2,\cdots,n.$$

The rest follows directly from estimate of Lemma A(see [9]).

In the following section we provide the reverse (necessity) in all parts of all theorems we formulated above. For that reason we again modify and generalize arguments that were proved in [8] recently.

4 Proofs of Theorems 1-4 (necessity of conditions)

Let us first show that conditions mentioned in polydisk in theorems we formulated above are necessary in Theorem 3 and Theorem 4. We will need some additional definitions. Let $R_{m_j} = \exp(-\frac{1}{2^{\lambda m_j}}), j = 1, 2, \cdots, n$. Let us assume that there is a function f from $N^4_{p,q,w}$ or $N^5_{p,q,w}$

$$\ln^+ \left| \frac{\partial^n f}{\partial z_1, \ldots, \partial z_n}(R_{m_1} e^{i\varphi_1}, \ldots, R_{m_n} e^{i\varphi_n}) \right| \geq C \sum_{j=1}^n \ln \frac{1}{1 - R_{m_j}}, f \in H(\mathbb{D}^n) \qquad (3)$$

First we show that this is enough for our purposes. First from last estimate we have the following inequalities.

$$\left(\ln^+ |\mathcal{D} f(R_{m_1} e^{i\varphi_1}, \ldots, R_{m_n} e^{i\varphi_n})| \right)^t \geq C \sum_{j=1}^n \ln(\frac{1}{1 - R_{m_j}})^t, t \in (0, \infty) \qquad (4)$$

and

$$\left(\int_{T^n} \ln^+ |\mathcal{D} f(R_{m_1} e^{i\varphi_1}, \ldots, R_{m_n} e^{i\varphi_n})| d\varphi_1 \cdots d\varphi_n \right)^p$$
$$\geq C \sum_{j=1}^n \ln \left(\frac{1}{1 - R_{m_j}} \right)^p, p \in (0, \infty) \quad (5)$$

For $R < 1, R_{N_1+1} > R, \ldots, R_{N_n+1} > R$. Then

$$\int_0^R \cdots \int_0^R G(\rho_1, \ldots, \rho_n) d\rho_1 \ldots d\rho_n$$
$$\leq C \sum_{K_1=1}^{N_1} \cdots \sum_{K_n=1}^{N_n} \left(\int_{R_{K_1}}^{R_{K_1+1}} \cdots \int_{R_{K_n}}^{R_{K_n+1}} G(\rho_1, \ldots, \rho_n) d\rho_1 \ldots d\rho_n \right)$$

On the other hand it is easy to show (see [8])

$$\ln^t \frac{1}{1 - R_{m_j}} \leq C \ln^t \frac{1}{1 - R_{m_j - 1}}, j = 1, 2, \cdots, n.$$

And hence we have combining estimates

$$\sum_{j=1}^n \int_0^R w_j(1 - \tau_j) \ln^s \frac{1}{1 - \tau_j} d\tau_j \prod_{i \neq j} \int_0^R w_i(1 - \tau_i) d\tau_i$$
$$\leq C \sum_{K_1=1}^{N_1} \cdots \sum_{K_n=1}^{N_n} \int_{T^n} \int_{R_{K_1}}^{R_{K_1+1}} \cdots \int_{R_{K_n}}^{R_{K_n+1}} \prod_{j=1}^n w_j(1 - \tau_j)(\ln^+ |\mathcal{D} f(\vec{\tau}\xi)|)^s d\xi_1 \ldots d\xi_n d\tau_1 \ldots d\tau_n.$$

We use the fact that $\int_{T^n} (\ln^+ |\mathcal{D} f(\vec{\tau}\xi)|)^s d\xi_1 \ldots d\xi_n$ is growing function for $s \geq 1$ and each $\tau_j, j = 1, 2, \cdots, n$. Passing to the limit when $R \to 1 - 0, N_j \to \infty, R_{n_j} = \exp(-\frac{1}{2^{\lambda n_j}}), \lambda > 0, j = 1, 2, \cdots, n$. We have

$$\begin{aligned}&\sum_{j=1}^{n}\int_0^1 w_j(1-\tau)\ln^s\frac{1}{1-\tau}d\tau\\ \leq\ & C\int_{T^n}\int_0^1\cdots\int_0^1\prod_{j=1}^{n}w_j(1-\tau)(\ln^+|\mathcal{D}f(\tau\xi)|)^s\tau d\tau d\xi_1\dots d\xi_n.\end{aligned}$$

This is what we need for Theorem 4.

Almost similarly modifying arguments from [8] we can show that (3) is enough for the proof Theorem 3 also using (5).

Indeed we have the following estimates. We have passing at final step to limit as $R\to 1-0, K_j\to\infty, j=1,2,\cdots,n$.

$$\begin{aligned}&\sum_{j=1}^{n}\int_0^R(\ln\frac{1}{1-R_j})^p w_j(1-R_j)dR_j\prod_{i\neq j}\int_0^R w_i(1-\tau)d\tau\\ \leq\ & C\sum_{j=1}^{n}\sum_{K_1=1}^{N_1}\cdots\sum_{K_n=1}^{N_n}\int_{R_{K_j}}^{R_{K_j+1}}(\ln\frac{1}{1-\tau})^p w_j(1-\tau)d\tau\\ &\prod_{i\neq j}\int_{R_{K_j}}^{R_{K_j+1}}w_i(1-\tau)d\tau, R_{N_j+1}>R,\quad j=1,2,\cdots,n.\end{aligned}$$

$$\sum_{j=1}^{n}\int_0^1\left(\ln\frac{1}{t}\right)^p w_j(t)dt\leq C\int_0^1\cdots\int_0^1 w_1(1-\tau)\cdots w_n(1-\tau)T^p(\mathcal{D}f,\vec{\tau})d\tau_1\dots d\tau_n.$$

Now let us show (3) estimate. For that reason we again modify and generalize arguments from [8]. Let $K_j=1,2,\dots,j=1,2,\cdots,n$. Let $\lambda_{k_j}=2^{\lambda k_j},\lambda>0$. Let $\alpha\in(0,1)$. Let also

$$\begin{aligned}f(z_1,z_2,\dots,z_n) &= \sum_{K_1=0}^{\infty}\lambda_{K_1}^{\alpha-1}z_1^{\lambda_{k_1}}\dots\sum_{K_n=0}^{\infty}\lambda_{K_n}^{\alpha-1}z_n^{\lambda_{k_n}}\\ &= \sum_{K_1=0}^{\infty}\dots\sum_{K_n=0}^{\infty}\lambda_{K_1}^{\alpha-1}\dots\lambda_{K_n}^{\alpha-1}z_1^{\lambda_{k_1}}\dots z_n^{\lambda_{k_n}}\\ &= \sum_{K_1=0}^{\infty}\dots\sum_{K_n=0}^{\infty}2^{\lambda K_1(\alpha-1)}\dots 2^{\lambda K_n(\alpha-1)}z_1^{\lambda_{k_1}}\dots z_n^{\lambda_{k_n}}\end{aligned}$$

$$\mathcal{D}f(z_1,z_2,\dots,z_n)=\sum_{K_1=0}^{\infty}\dots\sum_{K_n=0}^{\infty}2^{\lambda K_1\alpha}\dots 2^{\lambda K_n\alpha}z_1^{2^{\lambda K_1-1}}\dots z_n^{2^{\lambda K_n-1}}.$$

Since $\alpha<1$ it is not difficult to show that f belongs to classes we need. In the following argument we increase the number of variables in arguments provided in [8]. Note first $\frac{1}{1-\tau_n}=2^{\lambda n},\tau_n=\exp(-\frac{1}{2^{\lambda n}})$. Let $|z_j|=\exp(-\frac{1}{2^{\lambda m_j}})$, then we have the following

$$\mathcal{D}f(z)=\prod_{j=1}^{n}\sum_{K_j=0}^{\infty}2^{\lambda K_j\alpha}\exp[-\frac{1}{2^{\lambda m_j}}\cdot 2^{\lambda K_j-1}].$$

Let

$$U_{m_1,\ldots,m_n} = \prod_{j=1}^{n} 2^{\lambda\alpha\sum_{j=1}^{n} m_j} \exp[-\frac{1}{2^{\lambda m_j}} \cdot (2^{\lambda m_j} - 1)]$$

$$S_{m_1,\ldots,m_n} = \sum_{K_1=0}^{m_1-1} \cdots \sum_{K_n=0}^{m_n-1} 2^{\lambda\alpha\sum_{j=1}^{n} K_j} \cdot \prod_{j=1}^{n} \exp[-\frac{1}{2^{\lambda m_j}} \cdot (2^{\lambda K_j} - 1)]$$

$$R_{m_1,\ldots,m_n} = \prod_{j=1}^{n} \sum_{K_j=m_j+1}^{\infty} \exp[-\frac{1}{2^{\lambda m_j}} \cdot (2^{\lambda K_j} - 1)].$$

Then we have

$$R_{m_1,\ldots,m_n} + S_{m_1,\ldots,m_n} + U_{m_1,\ldots,m_n} \le |\mathcal{D}f|.$$

Following estimates from [8] by each variable we have the following estimates

$$U_{m_1,\ldots,m_n} = 2^{\lambda\alpha\sum_{j=1}^{n} K_j} \exp[-\frac{2^{\lambda m_1} - 1}{2^{\lambda m_1}}] \ldots \exp[-\frac{2^{\lambda m_n} - 1}{2^{\lambda m_n}}]$$

$$S_{m_1,\ldots,m_n} \le \frac{1}{(2^{\lambda\alpha} - 1)^n} 2^{\lambda\alpha(m_1+\ldots+m_n)}$$

$$R_{m_1,\ldots,m_n} \le (\frac{1}{\lambda \ln \lambda \exp(2^\lambda)})^n 2^{\lambda\alpha\sum_{j=1}^{n} m_j} \exp(\frac{1}{2^{\lambda m_1}}) \ldots \exp(\frac{1}{2^{\lambda m_n}})$$

Hence

$$|\mathcal{D}f| \ge |U_{m_1,\ldots,m_n}| - |S_{m_1,\ldots,m_n}| - |R_{m_1,\ldots,m_n}| \ge 2^{\lambda\alpha\sum_{j=1}^{n} m_j}(M_1(\lambda) - M_2(\lambda) - M_3(\lambda)).$$

We note again that estimates above $R_{m_1,\ldots,m_n}$ and $S_{m_1,\ldots,m_n}$ were obtained similarly as in [8] we simply increase the amount of variables in a standard way using one variable results n times.

It remains to make the following calculations see also [8].

$$\lim_{\lambda\to\infty} M_1(\lambda) = \lim_{\lambda\to\infty} \prod_{j=1}^{n} \exp(\frac{1}{2^{\lambda m_j}} - 1) = \frac{1}{e};$$

$$\lim_{\lambda\to\infty} M_2(\lambda) = \lim_{\lambda\to\infty} (\frac{1}{2^{\lambda\alpha} - 1})^n = 0;$$

$$\lim_{\lambda\to\infty} M_3(\lambda) = \lim_{\lambda\to\infty} \exp[\frac{1}{2^{\lambda m_1}} + \ldots + \frac{1}{2^{\lambda m_n}}] \cdot (\frac{1}{\lambda \ln \lambda \exp(2^\lambda)})^n = 0.$$

Hence we have finally

$$|\mathcal{D}f(\tau_{m_1} e^{i\varphi_1}, \ldots, \tau_{m_n} e^{i\varphi_n})| \ge C 2^{\lambda\alpha\sum_{j=1}^{n} m_j}, \lambda > \lambda_0, \varphi_i \in T, i = 1, 2, \ldots, m.$$

From here we get as in [8].

$$(\ln^+ |\mathcal{D}f(\tau_{m_1} e^{i\varphi_1}, \ldots, \tau_{m_n} e^{i\varphi_n})|)^t \ge C \sum_{j=1}^{n} (\ln \frac{1}{1 - \tau_{m_j}})^t, \varphi_i \in [0, 2\pi), t \in (0, \infty).$$

This is what we need.

Since Theorem 4 is a polydisk extension of Theorem 2 it remains to show the necessity of conditions of Theorem 1. For that reason we follows again the construction suggested in [8]. We had a function (see [8]),

$$f \in H(\mathbb{D}), f(z) = \sum_{k=0}^{\infty} \lambda_k^{\alpha-1} z^{\lambda_k}, 0 < \alpha < 1, \lambda_k = 2^{\lambda k}, \lambda > 0, \tau_n = \exp(-\frac{1}{\lambda n})$$

and

$$\ln^+ |f'(\tau_n e^{i\varphi})| \geq \left(C \ln \frac{1}{1-\tau_n}\right), \varphi \in [0, 2\pi). \tag{6}$$

Obviously $T(\tau, f) \leq C$. So we have to show that for this function the following estimates hold.

$$\sup_{R\in(0,1)} (1-R)^\alpha \int_0^R \left(\ln \frac{1}{1-\tau}\right)^p w(1-\tau) d\tau \leq C\|f'\|_{N^1_{p,w,\alpha}}, \alpha \geq 0, 0 < p < \infty;$$

$$\int_0^1 \sup_{R<\tau} w(1-R) \left(\ln \frac{1}{1-R}\right)^p (1-\tau)^\alpha d\tau \leq C\|f'\|_{N^2_{p,w,\alpha}}, \alpha > -1, 0 < p < \infty;$$

$$\int_0^1 \left(\int_0^R w(1-\tau) \left(\ln \frac{1}{1-\tau}\right)^p d\tau\right)^{\frac{q}{p}} (1-R)^\alpha dR \leq C\|f'\|_{N^3_{p,q,w,\alpha}}, \alpha > -1, 0 < p, q < \infty.$$

We provide a complete proofs of first and third estimate, the second one can be obtained by modification. From (6) we have

$$\begin{aligned} & \int_0^R \chi(\tau) w(1-\tau) \left(\ln \frac{1}{1-\tau}\right)^p d\tau \\ \leq & \sum_{n=1}^{N} \int_{\tau_{n-1}}^{\tau_n} \chi(\tau) w(1-\tau) \left(\ln \frac{1}{1-\tau}\right)^p d\tau \\ \leq & \sum_{n=1}^{N} \int_{\tau_{n-1}}^{\tau_n} \chi(\tau) w(1-\tau) \left(\ln \frac{1}{1-\tau_{n-1}}\right)^p d\tau, \tau_N > R, R < 1, \rho < 1 \end{aligned}$$

where $\chi = \chi(\tau) = \chi_{[0,\rho)}(\tau)$ is a characteristic function of $[0, \rho)$. From (6)

$$\begin{aligned} & \int_0^R \lambda(\tau) w(1-\tau) \left(\ln \frac{1}{1-\tau}\right)^p d\tau \\ \leq & \ C \sum_{n=1}^{N} \int_{\tau_{n-1}}^{\tau_n} \lambda(\tau) w(1-\tau) T^p(\tau_{n-1}, f'') d\tau. \end{aligned}$$

Hence passing to limit as $N \to +\infty, R \to (1-0)$ we have

$$\begin{aligned} & \sup_{\rho\in(0,1)} \int_0^\rho w(1-\tau) \left(\ln \frac{1}{1-\tau}\right)^p d\tau (1-\rho)^\alpha \\ & \qquad \leq \sup_{\rho\in(0,1)} \int_0^\rho w(1-\tau) T^p(\tau, f') d\tau (1-\rho)^\alpha \leq C\|f'\|_{N^1_{p,w,\alpha}}. \end{aligned}$$

$$\int_0^{\rho} w(1-\tau)\left(\ln\frac{1}{1-\tau}\right)^p d\tau \le C\int_0^{\rho} w(1-\tau)T^p(\tau,f')d\tau. \tag{7}$$

Hence we have

$$\sup_{\rho\in(0,1)}\int_0^{\rho} w(1-\tau)T^p(\tau,f')d\tau(1-\rho)^{\alpha} \le C\|f'\|_{N^1_{p,w,\alpha}}.$$

This is what was needed. Similarly from (7)

$$\int_0^1\left(\int_0^R w(1-\tau)\left(\ln\frac{1}{1-\tau}\right)^p d\tau\right)^{\frac{q}{p}}(1-R)^{\alpha}dR \le C\|f'\|_{N^3_{p,q,w,\alpha}}.$$

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